Exam Introduction to Mathematics Part 3:

Mathematical Modelling - Dimensional Analysis

November 1, 2013: 9.00-10.00.

This exam has 2 problems. Each problem is worth 5 points; more details can be found below. Write on each page your name and student number, and on the first page your seminar group. The use of annotations, books and calculators is not permitted in this examination. All answers must be supported by work. Success.

1. If we want to find the lift force on a frisbee we must first decide what might influence this force. Here we assume that the lift force F on a frisbee depends on the following physical variables: the diameter d of the frisbee, the number of revolutions per second, N, of the frisbee, the forward velocity u of the frisbee, the mass density ρ of the fluid and the fluid viscosity μ.

   (a) (*1 point*) These physical variables can be broken down into the fundamental dimensions length L, time T and mass M. In conjunction with this, we will designate the fundamental dimension of a physical variable x using the notation $[x]$. For example, $[\mu] = ML^{-1}T^{-1}$. Give $[F]$, $[d]$, $[N]$, $[u]$ and $[\rho]$.

   (b) (*4 points*) In mathematical terms, the modeling assumption is $F = f(d, N, u, \rho, \mu)$. Use dimensional analysis to show that this model can be reduced to

   $$\frac{F}{\rho d^2 u^2} = G\left(\frac{\mu}{du\rho}, \frac{Nd}{u}\right)$$

   where G is an arbitrary function (that is to be determined in a later stage).

2. From Newton's second law, the displacement $y(t)$ of the mass in a mass, spring, dashpot system satisfies

   $$m\frac{d^2y}{dt^2} = F_s + F_d$$

   for time $t > 0$, where m is the mass, F_s is the restoring force in the spring and F_d is the damping force. The initial condition for this problem are

   $$y(0) = 0, \qquad \frac{dy}{dt}(0) = v_0.$$

   (a) (*3 points*) Suppose there is no damping, so $F_d = 0$, and the spring is linear, so $F_s = -ky$. What are the dimensions of the spring constant k? Nondimensionalize the resulting initial value problem by introducing a change of variables, $t = t_c s$, $y = y_c u$, where t_c and y_c are constants that have the dimensions of time and displacement, respectively, and they represents a characteristic value of t and y, respectively. Your choice for t_c and y_c should result in no dimensionless products being left in the initial value problem.

(b) (*2 points*) Now, in addition, to a linear spring, suppose linear damping is included, so

$$F_d = -c\frac{dy}{dt}$$

What are the dimensions of the damping constant c? Using the same scaling as in part (a), nondimensionalize the problem. Your answer should contain a dimensionless parameter ϵ that measures the strength of the damping. In particular, if c is small then ϵ is small. The system in this case is said to have weak damping.